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Clifford Deformations of Koszul Frobenius Algebras and Noncommutative Quadrics

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Abstract. A Clifford deformation of a Koszul Frobenius algebra E is a finite dimensional $\mathbb{Z}_2$-graded algebra $E(\theta)$, which corresponds to a noncommutative quadric hypersurface $E^!/(z)$ for some central regular element $z \in E^!_2$. It turns out that the bounded derived category $D^{\mathrm{b}}(\mathrm{gr}_{\mathbb{Z}_2} E(\theta))$ is equivalent to the stable category of the maximal Cohen-Macaulay modules over $E^!/(z)$ provided that $E^!$ is noetherian. As a consequence, $E^!/(z)$ is a noncommutative isolated singularity if and only if the corresponding Clifford deformation $E(\theta)$ is a semisimple $\mathbb{Z}_2$-graded algebra. The preceding equivalence of triangulated categories also indicates that Clifford deformations of trivial extensions of a Koszul Frobenius algebra are related to Knörrer's periodicity theorem for quadric hypersurfaces. As an application, we recover Knörrer's periodicity theorem without using matrix factorizations.

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1 Introduction

Let S be a noetherian Koszul Artin-Schelter regular algebra, and let $z \in S_2$ be a central regular element of S. The quotient algebra $A = S/(z)$ is a Koszul Artin-Schelter Gorenstein algebra, which is usually called a quadric hypersurface algebra. Smith and Van den Bergh introduced in [20] a finite dimensional algebra $C(A)$ associated to the quadric hypersurface algebra A, which determines the representations of the singularities of A. In particular, the simplicity of $C(A)$ implies the smoothness of $\operatorname{Proj} A$ (see [20, Proposition 5.2]), where $\operatorname{Proj} A$ is the noncommutative projective scheme (see [2]).

Let $E = S^!$ be the quadratic dual algebra of S. Then E is a Koszul Frobenius algebra. The dimension of $C(A)$ is equal to that of the even degree part of E (see [20, Lemma 5.1]). As the key observation of this paper, we notice that $C(A)$ may be obtained from some Poincaré-Birkhoff-Witt (PBW) deformation of E.

Write the above Koszul Frobenius algebra E as $E = T(V)/(R)$ for some finite dimensional vector space V over a field $\Bbbk$ and $R \subseteq V \otimes V$. Let $\theta\colon R \to \Bbbk$ be a linear map. If $(\theta \otimes 1 - 1 \otimes \theta)(V \otimes R \cap R \otimes V) = 0$, then θ defines a PBW deformation $E(\theta)$ of E (see [15, Proposition 5.1.1]); that is, $E(\theta)$ is a filtered algebra whose associated graded algebra is isomorphic to E. We call θ a *Clifford map* and $E(\theta)$ a *Clifford deformation* of E (more precisely, see Definition 3.1). The first examples are classical Clifford algebras, which motivate the name of Clifford deformation. In fact, if we take E to be the exterior algebra generated by V, then any Clifford map corresponds to a symmetric bilinear form on V, and the associated Clifford deformation can realize the classical Clifford algebras (see Example 3.3).

For every Clifford map θ, the algebra $E(\theta)$ has a natural $\mathbb{Z}_2$-graded structure. Note that every Clifford map θ corresponds to a central element $z \in S_2$ (see Remark 3.10). It turns out that the degree 0 part $E(\theta)_0$ coincides with the finite dimensional algebra $C(A)$ if z is regular (see Proposition 5.3). Hence, the structure of $E(\theta)$ will determine the representations of singularities of A.

Let $\mathrm{gr}_{\mathbb{Z}_2} E(\theta)$ be the category of finite dimensional right $\mathbb{Z}_2$-graded $E(\theta)$-modules. Let $\underline{\mathrm{mcm}}\, A$ be the stable category of graded maximal Cohen-Macaulay modules over A. The main observation of the paper is the following result (cf. Theorem 6.1(iii)).

Theorem 1.1. *Let S be a noetherian Koszul Artin-Schelter regular algebra and let $z \in S_2$ be a central regular element. Set $A = S/(z)$ and let $E = S^!$ be the quadratic dual algebra of S. Assume that θ_z is the Clifford map corresponding to z. Then there is an equivalence of triangulated categories $D^{\mathrm{b}}(\mathrm{gr}_{\mathbb{Z}_2} E(\theta_z)) \cong \underline{\mathrm{mcm}}\, A$.*

Let $\mathrm{gr}\, A$ be the category of finitely generated right graded A-modules , and $\mathrm{tor}\, A$ be the full subcategory consisting of finite dimensional ones. Set $\mathrm{qgr}\, A = \mathrm{gr}\, A/\mathrm{tor}\, A$ to be the quotient category, which is again an abelian category. If $\mathrm{qgr}\, A$ has finite global dimension, then A is called a *noncommutative isolated singularity* (see [21]), or equivalently, $\mathrm{Proj}\, A$ is *smooth* (see [20]).

Let S, z and A be as above. Assume $\mathrm{gldim}(S) \geq 2$. In [20, Proposition 5.2(2)], Smith and Van den Bergh showed that if $C(A)$ is semisimple, then $\mathrm{Proj}\, A$ is smooth. Moreover, they showed that the converse is also true in some special case (see [20, Theorem 5.6]). It is natural to ask whether the converse statement holds true in general. By applying Theorem 1.1, we have the following result, which essentially gives an affirmative answer to this question (cf. Theorem 7.3).

Theorem 1.2. *Retain the notation of Theorem* 1.1 *and assume* $\mathrm{gldim}(S) \geq 2$. *Then A is a noncommutative isolated singularity if and only if $E(\theta_z)$ is a semisimple $\mathbb{Z}_2$-graded algebra.*

Note that the sufficiency part of the above theorem is essentially a consequence of [20, Proposition 5.2(2)]. For the necessity part, Smith and Van den Bergh have also shown a special case, say if S has Hilbert series $(1-t)^{-4}$ (see [20, Theorem

5.6]). In this case the corresponding algebra $C(A)$ is of dimension 8, and the proof depends on a detailed analysis of the representations of $C(A)$. We mention that the method we use to prove the necessity part of Theorem 1.2 is totally different from that of [20, Theorem 5.6], and our method works for general Koszul Artin-Schelter regular algebras.

Compared to the algebra $C(A)$ constructed in [20], the Clifford deformations of a Koszul Frobenius algebra are relatively easy to determine. In particular, when S has lower global dimension, it is possible to find all the quadric hypersurfaces obtained from S and determine whether they are isolated singularities or not, by a detailed analysis on the possible structures of Clifford deformations $E(\theta)$ (see Section 10).

Another advantage of Clifford deformations is that they can give a new explanation of Knörrer's periodicity theorem (cf. [7, Theorem 3.1], and the noncommutative case [4, Theorem 1.7]), at least for quadric hypersurfaces. The methods used in [7] and in [4] to prove Knörrer's periodicity theorem are matrix factorizations (more generally, see [11]). In our observation, Knörrer's periodicity theorem for quadric hypersurfaces may be explained by Clifford deformations of trivial extensions of Koszul Frobenius algebras (see Section 8).

Let $\tilde{E} := E \oplus {}_{\epsilon}E(-1)$ be the trivial extension of E (precisely, see Section 8). Then $\tilde{E}$ is also a Koszul Frobenius algebra. A Clifford map θ of E induces a Clifford map $\tilde{\theta}$ of $\tilde{E}$. Iterating the trivial extensions, we obtain a Koszul algebra $\tilde{\tilde{E}}$ and a Clifford map $\tilde{\tilde{\theta}}$ of $\tilde{\tilde{E}}$. Assume that the base field $\Bbbk = \mathbb{C}$. Then we have the following periodicity property (cf. Proposition 8.9).

Proposition 1.3. *There is an equivalence* $\operatorname{gr}_{\mathbb{Z}_2} \tilde{\tilde{E}}(\tilde{\tilde{\theta}}) \cong \operatorname{gr}_{\mathbb{Z}_2} E(\theta)$ *of abelian categories.*

Retain the notation of Theorem 1.1 and assume $\Bbbk$ is the field of complex numbers. A *double branched cover* of A is defined to be the Artin-Schelter Gorenstein algebra $A^{\#} = S[v]/(z + v^2)$ and the *second double branched cover* of A is defined to be the Artin-Schelter Gorenstein algebra $A^{\#\#} = S[v_1, v_2]/(z + v_1^2 + v_2^2)$ (see [8, Chapter 8] and [4]). The above periodicity property of Clifford deformations of iterated trivial extensions of Koszul Frobenius algebras implies the following Knörrer periodicity theorem for noncommutative quadric hypersurfaces (cf. Theorems 9.1 and 9.2). We remark that similar results were also obtained in [12] recently by using noncommutative matrix factorizations.

Theorem 1.4. *Retain the notation as above. Assume that* $\operatorname{gldim} S \geq 2$.
(i) *A is a noncommutative isolated singularity if and only if so is $A^{\#}$.*
(ii) *There is an equivalence of triangulated categories* $\underline{\operatorname{mcm}}\, A \cong \underline{\operatorname{mcm}}\, A^{\#\#}$.

For a concrete Koszul Artin-Schelter regular algebra S, Clifford deformations of the quadratic dual $E = S^!$ are relatively easy to determine. In the last section, we give detailed computations of Clifford deformations for the quadratic dual algebra of a concrete Koszul Artin-Schelter regular algebra S of global dimension 3, and then we give a list of all the possible noncommutative quadric hypersurfaces associated to S (up to isomorphism). We also determine whether the obtained noncommutative hypersurfaces are isolated singularities or not.

2 Preliminaries

Let $\Bbbk$ be a field of characteristic zero. A $\mathbb{Z}$-graded algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is called a *connected graded algebra* if $A_n = 0$ for $n < 0$ and $A_0 = \Bbbk$. Let $\operatorname{Gr} A$ denote the category whose objects are right graded A-modules and whose morphisms are right A-module morphisms that preserve the gradings of modules. Let $\operatorname{gr} A$ denote the full subcategory of $\operatorname{Gr} A$ consisting of finitely generated graded A-modules. For a right graded A-module M and an integer l, we write $M(l)$ for the right graded A-module whose ith part is $M(l)_i = M_{i+l}$.

Define $\underline{\operatorname{Hom}}_A(M, N) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr} A}(M, N(i))$ for right graded A-modules M and N. Then $\underline{\operatorname{Hom}}_A(M, N)$ is a $\mathbb{Z}$-graded vector space. Write $\underline{\operatorname{Ext}}^i_A$ for the ith derived functor of $\underline{\operatorname{Hom}}_A$. Hence, $\underline{\operatorname{Ext}}^i_A(M, N)$ is also a $\mathbb{Z}$-graded vector space for each $i \geq 0$. We mention that if A is noetherian and M is finitely generated, then $\underline{\operatorname{Ext}}^i_A(M, N) \cong \operatorname{Ext}^i_A(M, N)$, the usual extension group in the category of right A-modules.

Definition 2.1. [1] A connected graded algebra A is called an *Artin-Schelter Gorenstein algebra* of injective dimension d if

(i) A has finite injective dimension $\operatorname{injdim}{}_A A = \operatorname{injdim} A_A = d < \infty$,

(ii) $\underline{\operatorname{Ext}}^i_A(\Bbbk, A) = 0$ for $i \neq d$, and $\underline{\operatorname{Ext}}^d_A(\Bbbk, A) \cong \Bbbk(l)$, and

(iii) the left version of (ii) holds.

The number l is called the *Gorenstein parameter*. If further, A has finite global dimension, then A is called an *Artin-Schelter regular* algebra.

We need the following lemma, which follows from the Rees lemma (see [9, Proposition 3.4(b)]), or the base-change for the spectral sequence [23, Exercise 5.6.3].

Lemma 2.2. *Let A be an Artin-Schelter regular algebra of global dimension d with Gorenstein parameter l. Let $z \in A_k$ be a central regular element of A. Then A/Az is an Artin-Schelter Gorenstein algebra of injective dimension $d-1$ with Gorenstein parameter $l-k$.*

A locally finite connected graded algebra A is called a *Koszul algebra* (see [17]) if the trivial module $\Bbbk_A$ has a free resolution $\cdots \to P^n \to \cdots \to P^1 \to P^0 \to \Bbbk \to 0$, where P^n is a graded free module generated in degree n for each $n \geq 0$. Locally finite means that each A_i is of finite dimension. A Koszul algebra is a quadratic algebra in the sense that $A \cong T(V)/(R)$, where V is a finite dimensional vector space and $R \subseteq V \otimes V$. For a Koszul algebra A, the *quadratic dual* of A is the quadratic algebra $A^! = T(V^*)/(R^\perp)$, where V^* is the dual vector space and $R^\perp \subseteq V^* \otimes V^*$ is the orthogonal complement of R. Note that $A^!$ is also a Koszul algebra.

A $\mathbb{Z}$-graded finite dimensional algebra E is called a *graded Frobenius* algebra if there is an isomorphism of right graded E-modules $E \cong E^*(l)$ for some $l \in \mathbb{Z}$, or equivalently, there is a nondegenerate associative bilinear form $\langle -,- \rangle \colon E \times E \to \Bbbk$ such that for homogeneous elements $a \in E_i$ and $b \in E_j$, $\langle a, b\rangle = 0$ if $i + j \neq l$. We will freely use the following connections between graded Frobenius algebras and Koszul Artin-Schelter regular algebras.

Lemma 2.3. [19, Proposition 5.10] *Let A be a Koszul algebra and let $A^!$ be its quadratic dual. Then A is Artin-Schelter regular if and only if $A^!$ is a graded Frobenius algebra.*

Convention. In this paper, unless otherwise stated, a graded algebra always means a $\mathbb{Z}$-graded algebra.

3 Clifford Deformations of Koszul Frobenius Algebras

Let V be a finite dimensional vector space and $R \subseteq V \otimes V$. Now we suppose that $E = T(V)/(R)$ is a Koszul Frobenius algebra.

Definition 3.1. Let $\theta\colon R \to \Bbbk$ be a linear map. If $(\theta\otimes 1 - 1\otimes\theta)(V\otimes R \cap R\otimes V) = 0$, then we call θ a *Clifford map* of the Koszul Frobenius algebra E, and call the associative algebra $E(\theta) = T(V)/(r - \theta(r) : r \in R)$ the *Clifford deformation* of E associated to θ.

We view $T(V)$ as a filtered algebra with the following filtration: $F_iT(V) = 0$ for $i < 0$, $F_iT(V) = \sum_{j=0}^{i} V^{\otimes j}$ for $i \geq 0$. The filtration on $T(V)$ induces a filtration

$$0 \subseteq F_0E(\theta) \subseteq F_1E(\theta) \subseteq \cdots \subseteq F_iE(\theta) \subseteq \cdots \tag{1}$$

on $E(\theta)$ making $E(\theta)$ a filtered algebra. Let $\mathsf{gr}\, E(\theta)$ be the graded algebra associated to the filtration (1). The next result is a special case of [15, Theorem 5.2.1].

Proposition 3.2. *Let $E(\theta)$ be a Clifford deformation of E. Then $\mathsf{gr}\, E(\theta) \cong E$ as graded algebras.*

For later use, we define a linear transformation $\mathfrak{c}$ of the algebra $T(V)$ by setting $\mathfrak{c}(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_i \otimes \cdots \otimes v_n \otimes v_1 \otimes \cdots \otimes v_{i-1}$ for $n \geq 2$ and $v_1, \ldots, v_n \in V$, $\mathfrak{c}(v) = v$ for all $v \in V$, and $\mathfrak{c}(1) = 1$.

Example 3.3. Let V be a finite dimensional vector space over the field of real numbers $\mathbb{R}$ with a basis $\{x_1, \ldots, x_n\}$. Consider the exterior algebra $E = \wedge V$. Then $E = T(V)/(R)$, where R is the subspace of $V \otimes V$ spanned by $x_ix_j + x_jx_i$ (for simplicity, we omit the notation $\otimes$) for all $1 \leq i, j \leq n$. Then $V \otimes R \cap R \otimes V$ admits a basis: x_i^3 $(i = 1, \ldots, n)$, $\mathfrak{c}(x_i^2x_j)$ for $i \neq j$ $(i, j = 1, \ldots n)$, $\mathfrak{c}(x_ix_jx_k + x_jx_ix_k)$ for pairwise different triples (i, j, k) $(i, j, k = 1, \ldots, n)$. Define a linear map $\theta\colon R \to \Bbbk$ by setting $\theta(x_i^2) = -1$ for $1 \leq i \leq p$, $\theta(x_i^2) = 1$ for $p+1 \leq i \leq p+q$ with $p+q \leq n$, $\theta(x_i^2) = 0$ for $i > p+q$, and $\theta(x_ix_j + x_jx_i) = 0$ for $i \neq j$. Then it is easy to see that θ is a Clifford map of E, and $E(\theta)$ is a Clifford deformation of the exterior algebra $\wedge V$. In fact, $E(\theta)$ is isomorphic to the Clifford algebra $\mathbb{R}^{p,q}$ (see [16, Proposition 15.5]) defined by the quadratic form $\rho(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$.

Example 3.4. Let $V = \Bbbk x \oplus \Bbbk y$ and $E = T(V)/(R)$, where $R = \operatorname{span}\{x^2, y^2, xy - yx\}$. Define a map $\theta\colon R \to \Bbbk$ by setting $\theta(x^2) = a$, $\theta(y^2) = b$ and $\theta(xy - yx) = 0$, where a, b are arbitrary elements in $\Bbbk$. Note that

$$V \otimes R \cap R \otimes V = \operatorname{span}\{x^3,\ y^3,\ x^2y - xyx + yx^2,\ xy^2 - yxy + y^2x\} \subseteq V \otimes V \otimes V.$$

Then it is easy to check $(\theta\otimes 1-1\otimes\theta)(V\otimes R\cap R\otimes V)=0$. Hence, θ is a Clifford map of E and $E(\theta)$ is a Clifford deformation of E.

Example 3.5. Let $V=\Bbbk x\oplus\Bbbk y$ and let $E=\wedge V$ be the exterior algebra. Then the generating relations of E are $R=\operatorname{span}\{x^2,y^2,xy+yx\}$. Define a map $\theta\colon R\to\Bbbk$ by setting $\theta(x^2)=a$, $\theta(y^2)=b$ and $\theta(xy+yx)=c$, where a,b,c are arbitrary elements in $\Bbbk$. By Example 3.3, $V\otimes R\cap R\otimes V=\operatorname{span}\{x^3,\ y^3,\ x^2y+xyx+yx^2,\ xy^2+yxy+y^2x\}$. It is not hard to check that θ is a Clifford map of $\wedge V$.

Example 3.6. Let $V=\Bbbk x\oplus\Bbbk y\oplus\Bbbk z$, and let $E=T(V)/(R)$, where R is the subspace of $V\otimes V$ spanned by $xz-zx$, $yz-zy$, x^2-y^2, z^2, xy, yx. Then E is a Koszul Frobenius algebra. Indeed, it is the quadratic dual algebra of the Artin-Schelter regular algebra of type S_2 (see [1, Table 3.11]). One may check that $\dim(V\otimes R\cap R\otimes V)=10$ and $V\otimes R\cap R\otimes V$ has the following basis:

$$\begin{aligned}
&z^3=z^3,\quad yxy=yxy,\quad xyx=xyx,\\
&(x^2z-xzx)-(y^2z-yzy)+(zx^2-zy^2)\\
&\quad=-(xzx-zx^2)+(yzy-zy^2)+(x^2z-y^2z),\\
&(xyz-xzy)+zxy=-(xzy-zxy)+xyz,\\
&(x^3-xy^2)-y^2x=(x^3-y^2x)-xy^2,\\
&(yxz-yzx)+zyx=-(yzx-zyx)+yxz,\\
&xz^2-(zxz-z^2x)=z^2x+(xz^2-zxz),\\
&yz^2-(zyz-z^2y)=z^2y+(yz^2-zyz),\\
&x^2y+(yx^2-y^3)=yx^2+(x^2y-y^3),
\end{aligned}$$

where the elements on the left-hand side are written as elements in $V\otimes R$, and the elements on the right-hand side in $R\otimes V$. Define a map $\theta\colon R\to\Bbbk$ by setting $\theta(z^2)=1$, $\theta(xy)=1$, $\theta(yx)=1$, $\theta(x^2-y^2)=1$ and $\theta(\text{others})=0$. It is not hard to check that θ is a Clifford map of E, and hence $E(\theta)$ is a Clifford deformation of E.

The next example shows that not every Koszul Frobenius algebra admits a nontrivial Clifford deformation.

Example 3.7. Let $V=\Bbbk x\oplus\Bbbk y$ and $E=T(V)/(R)$, $R=\operatorname{span}\{y^2,\ xy+yx,\ yx+x^2\}$. Then E is a Koszul Frobenius algebra, which is the quadratic dual algebra of a Jordan plane (see [1, Introduction]). The space $V\otimes R\cap R\otimes V$ admits a basis $\{y^3,\ xy^2+yxy+y^2x,\ 2y^2x+xyx+yx^2+yxy+x^2y,\ xyx+x^3+2y^2x+2yx^2\}$. Let $\theta\colon R\to\Bbbk$ be a Clifford map defined by $\theta(y^2)=a$, $\theta(xy+yx)=b$, $\theta(yx+x^2)=c$. Then the equations $(\theta\otimes 1-1\otimes\theta)(2y^2x+xyx+yx^2+yxy+x^2y)=2ax-by=0$ and $(\theta\otimes 1-1\otimes\theta)(xyx+x^3+2y^2x+2yx^2)=(2a+b)x-2cy=0$ imply $a=b=c=0$.

Clifford maps of a Koszul Frobenius algebra are corresponding to the central elements of degree 2 of its quadratic dual algebra. Let $E=T(V)/(R)$ be a Koszul Frobenius algebra. Let $E^!=T(V^*)/(R^\perp)$ be the quadratic dual algebra of E. We may identify R^* with $E^!_2$. The next lemma is a special case of [15, Proposition 5.4.1].

Lemma 3.8. *Retain the notation as above. A linear map $\theta\colon R \to \Bbbk$ is a Clifford map of E if and only if θ, viewed as an element in $E^!_2$, is a central element of $E^!$.*

Since $E^!_2 \cong R^*$, the above lemma shows that the set of the Clifford maps of E is in one-to-one correspondence with the set of the central elements in $E^!_2$.

By Lemma 2.3, every Koszul Frobenius algebra is dual to a Koszul Artin-Schelter regular algebra. Let $S = T(U)/(R)$ be a Koszul Artin-Schelter regular algebra, and let $E := S^! = T(U^*)/(R^\perp)$. Denote by $\pi_S\colon T(U) \to S$ the natural projection map.

Let $z \in S_2$ be a central element of S. Pick an element $r_0 \in U \otimes U$ such that $\pi_S(r_0) = z$. Since $R^\perp \subseteq U^* \otimes U^*$, the element r_0 defines a map

$$\theta_z\colon R^\perp \longrightarrow \Bbbk \quad \text{by} \quad \theta_z(\alpha) = \alpha(r_0) \;\; \forall\, \alpha \in R^\perp. \tag{2}$$

Lemma 3.8 implies the following result.

Lemma 3.9. *Retain the notation as above. The map $\theta_z\colon R^\perp \to \Bbbk$ is a Clifford map of the Koszul Frobenius algebra $E = T(U^*)/(R^\perp)$.*

Remark 3.10. Note that the Clifford map θ_z is independent of the choice of r_0. In fact, if $r_0' \in U \otimes U$ is another element such that $\pi_S(r_0') = z$, then for every element $\alpha \in R^\perp$ one has $\alpha(r_0) = \alpha(r_0')$. Henceforth, we say that θ_z is the *Clifford map of* $E\,(= S^!)$ *corresponding to the central element* z.

4 Clifford Deformations as $\mathbb{Z}_2$-Graded Frobenius Algebras

Let G be a group and $A = \bigoplus_{g\in G} A_g$ a G-graded algebra. Set $A^* := \bigoplus_{g\in G} (A_{g^{-1}})^*$. Then A^* is a G-graded A-bimodule, whose degree g component is $(A_{g^{-1}})^*$. Let M be a left G-graded A-module. For $g \in G$, let $M(g)$ be the left G-graded A-module whose degree h component $M(g)_h$ is equal to M_{hg}.

Similarly to $\mathbb{Z}$-graded Frobenius algebras, a G-graded algebra A is called a *G-graded Frobenius algebra* (see [5] for instance) if there is an element $g \in G$ and an isomorphism $\varphi\colon A \to A^*(g)$ of left G-graded A-modules. Equivalently, there is a homogenous bilinear form $\langle \text{-}, \text{-} \rangle\colon A \times A \to \Bbbk(g)$ such that $\langle ab, c\rangle = \langle a, bc\rangle$, where $\Bbbk(g)$ is the G-graded vector space concentrated in degree g^{-1}.

Let $E = T(V)/(R)$ be a Koszul Frobenius algebra. Let $\theta\colon R \to \Bbbk$ be a Clifford map of E. We may view the $\mathbb{Z}$-graded algebra $T(V)$ as a $\mathbb{Z}_2$-graded algebra by setting $T(V)_0 = \Bbbk \oplus \bigoplus_{n\geq 1} V^{\otimes 2n}$ and $T(V)_1 = \bigoplus_{n\geq 1} V^{\otimes 2n-1}$.

Consider the Clifford deformation $E(\theta) = T(V)/(r - \theta(r) : r \in R)$. Since $R \subseteq V \otimes V$, it follows that $R_\theta = \operatorname{span}\{r - \theta(r) \mid r \in R\}$ is a subspace of $T(V)_0$. Hence, the ideal (R_θ) is homogeneous. Therefore, we have the following observation.

Lemma 4.1. *Retain the notation as above. The Clifford deformation $E(\theta)$ is a $\mathbb{Z}_2$-graded algebra.*

Remark 4.2. Note that the filtration (1) induces a homomorphism $\psi\colon \mathsf{gr}\, E(\theta) \to E$ of $\mathbb{Z}$-graded algebras. By Proposition 3.2, ψ is an isomorphism. By the definition of the $\mathbb{Z}_2$-grading of $E(\theta)$, we observe that $\dim E(\theta)_0 = \dim\big(\bigoplus_{i\geq 0} E_{2i}\big)$ and $\dim E(\theta)_1 = \dim\big(\bigoplus_{i\geq 0} E_{2i+1}\big)$.

Assume that the Frobenius algebra E is of *Loevy length* n, that is, $E_n \neq 0$ and $E_i = 0$ for all $i > n$. Since E is $\mathbb{Z}$-graded Frobenius, $\dim E_n = 1$. Fix a nonzero map $\xi\colon E_n \to \Bbbk$. Let $\phi\colon E(\theta) \to \Bbbk$ be the composition of the following maps:

$$\phi:\ E(\theta) \xrightarrow{\pi} E(\theta)/F_{n-1}E(\theta) \xrightarrow{\psi} E_n \xrightarrow{\xi} \Bbbk, \tag{3}$$

where $F_{n-1}E(\theta)$ is the $(n-1)$th part in the filtration (1) and π is the projection. Define a bilinear form $\langle\text{-},\text{-}\rangle\colon E(\theta) \times E(\theta) \to \Bbbk$ by setting

$$\langle a, b\rangle = \phi(ab) \quad \text{for all } a, b \in E(\theta). \tag{4}$$

Lemma 4.3. *Retain the notation as above. The bilinear form $\langle\text{-},\text{-}\rangle$ is nondegenerated, and hence $E(\theta)$ is a Frobenius algebra.*

Proof. For a nonzero element $a \in E(\theta)$, assume that $a \in F_iE(\theta)$ but $a \notin F_{i-1}E(\theta)$. Write $\overline{a}$ for the corresponding element in $F_iE(\theta)/F_{i-1}E(\theta)$. Then $\overline{a} \neq 0$. Since E is a Koszul Frobenius algebra, there is an element $b' \in E_{n-i}$ such that $\psi(\overline{a})b' \neq 0$. Let $\overline{b} = \psi^{-1}(b') \in F_{n-i}E(\theta)/F_{n-i-1}E(\theta)$. Pick an element $b \in F_{n-i}E(\theta)$ corresponding to the element $\overline{b} \in F_{n-i}E(\theta)/F_{n-i-1}E(\theta)$. Then $\langle a, b\rangle = \phi(ab) = \xi\psi\pi(ab) = \xi\psi(\overline{a}\overline{b}) = \xi(\psi(\overline{a})\psi(\overline{b})) = \xi(\psi(\overline{a})b') \neq 0$. Similarly, there is an element $c \in E(\theta)$ such that $\langle c, a\rangle \neq 0$. Hence, $\langle\text{-},\text{-}\rangle$ is nondegenerated. □

Remark 4.4. That $E(\theta)$ is self-injective follows from a more general theory. In fact, $\mathsf{gr}\, E(\theta)$ is a Frobenius algebra; then by [10, Lemma I.6.11 and Theorem I.6.12], $E(\theta)$ is self-injective. Lemma 4.3 above shows that the bilinear form on $E(\theta)$ is indeed inherited from the bilinear form of E.

Proposition 4.5. *Retain the notation as above. The bilinear form defined in* (4) *is compatible with the $\mathbb{Z}_2$-grading of $E(\theta)$, and hence $E(\theta)$ is a $\mathbb{Z}_2$-graded Frobenius algebra.*

Proof. Assume that the Loevy length of E is n. Note that the filtration (1) is compatible with the $\mathbb{Z}_2$-grading. Then the map ϕ defined by (3) is homogeneous, that is, $\phi\colon E(\theta) \to \Bbbk(g)$, where $g \in \mathbb{Z}_2$ such that $g = 1$ if n is odd, or $g = 0$ if n is even. Since the $\langle\text{-},\text{-}\rangle$ is the composition of the multiplication of $E(\theta)$ and the map ϕ, we get $\langle\text{-},\text{-}\rangle\colon E(\theta) \times E(\theta) \to \Bbbk(g)$ with $g = 1$ if n is odd or $g = 0$ if n is even. □

Recall from the definition of the $\mathbb{Z}_2$-grading on $E(\theta)$ that $E(\theta)_0$ is the quotient space of $\Bbbk \oplus \bigoplus_{k\geq 1} V^{\otimes 2k}$. Since the generating relations R_θ of $E(\theta)$ are concentrated in degree 0 part, we may view V as a subspace of $E(\theta)_1$. Hence, each element $a \in E(\theta)_0$ may be written as $a = b + k$, where b is a sum of products of elements in V and $k \in \Bbbk$. Now assume that the Clifford map θ is nontrivial, and assume that $\theta(r) = k \neq 0$ for some $r \in R \subseteq V \otimes V$. Suppose $r = \sum_{i=1}^{l} x_i \otimes y_i$. Then $\sum_{i=1}^{l} x_i y_i = k$ in $E(\theta)_0$. Therefore, $E(\theta)_0 = E(\theta)_1 E(\theta)_1$. In summary, we have the following result. Recall that a G-graded algebra $B = \bigoplus_{g\in G} B_g$ is said to be *strongly graded* [13] if $B_g B_h = B_{gh}$ for all $g, h \in G$.

Proposition 4.6. *If the Clifford map θ is nontrivial, then the $\mathbb{Z}_2$-graded algebra $E(\theta)$ is a strongly graded algebra.*

5 Clifford Deformations from Localizations

Let E be a Koszul Frobenius algebra, and B a graded algebra generated in degree 1. Assume that B is a graded extension of E by a central regular element of degree 2; i.e., B is a quadratic graded algebra and there is a central regular element $z \in B_2$ such that $E = B/Bz$. It follows that the degree 1 part of E and B are equal. Since E is a Koszul algebra, B is also a Koszul algebra (see [18, Theorem 1.2]).

Assume $B = T(V)/(R)$ with $R \subseteq V \otimes V$. Let $\pi\colon T(V) \to B$ be the projection map. Pick an element $r_0 \in V \otimes V$ such that $\pi(r_0) = z$. Since $E = B/Bz$, it follows that $E = T(V)/(\Bbbk r_0 + R)$. Let $R' = \Bbbk r_0 \oplus R$. Define a map

$$\theta\colon R' \longrightarrow \Bbbk \quad \text{by} \quad r_0 \longmapsto 1, \quad r \longmapsto 0 \ \text{ for all } r \in R. \tag{5}$$

Let us check the elements in $R' \otimes V \cap V \otimes R'$. Assume that $\{r_1, \dots, r_t\}$ is a basis of R. For each element $\alpha \in R' \otimes V \cap V \otimes R'$, we have

$$\alpha = v_0 \otimes r_0 + \sum_{i=1}^{t} v_i \otimes r_i = r_0 \otimes v_0' + \sum_{i=1}^{t} r_i \otimes v_i', \quad v_0, \dots, v_t, v_0', \dots, v_t' \in V. \tag{6}$$

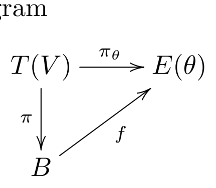
Lemma 5.1. *In Equation* (6), $v_0 = v_0'$.

Proof. Assume that $v_0' = v_0 + u$ for some $u \in V$. Then by Equation (6) we have $v_0 \otimes r_0 + \sum_{i=1}^{t} v_i \otimes r_i = r_0 \otimes v_0 + r_0 \otimes u + \sum_{i=1}^{t} r_i \otimes v_i'$. Hence, it follows that $r_0 \otimes v_0 - v_0 \otimes r_0 + r_0 \otimes u = \sum_{i=1}^{t} v_i \otimes r_i - \sum_{i=1}^{t} r_i \otimes v_i'$. Since the right-hand side of the above equation lies in $R \otimes V + V \otimes R$, we have the following identity in the algebra B: $\pi(r_0)\pi(v_0) - \pi(v_0)\pi(r_0) - \pi(r_0)\pi(u) = 0$. Note that $z = \pi(r_0)$ and z is a central regular element in B. It follows that $\pi(u) = 0$. Since π is injective when it is restricted to V, it follows that $u = 0$. □

Lemma 5.2. *Retain the above notation. The map θ in* (5) *is a Clifford map of E.*

Proof. By Lemma 5.1, each element $\alpha \in V \otimes R' \cap R' \otimes V$ may be written as $\alpha = v_0 \otimes r_0 + \sum_{i=1}^{t} v_i \otimes r_i = r_0 \otimes v_0 + \sum_{i=1}^{t} r_i \otimes v_i'$ for some $v_0, \dots, v_t, v_1', \dots, v_t' \in V$. Then it is easy to see that $(\theta \otimes 1 - 1 \otimes \theta)(\alpha) = 0$. □

Let $\tilde{R} = \Bbbk(r_0 - 1) \oplus R$. Then the Clifford deformation of E may be written as $E(\theta) = T(V)/(\tilde{R})$. Since $B = T(V)/(R)$, we have a natural algebra morphism $f\colon B \to E(\theta)$ such that the diagram



commutes, where π_θ is the projection map. Then $f(z) = f(\pi(r_0)) = \pi_\theta(r_0) = 1$.

Note that z is a central regular element of B. Let $B[z^{-1}]$ be the localization of B by the multiplicative set $\{1, z, z^2, \dots\}$. Since $f\colon B \to E(\theta)$ is an algebra morphism such that $f(z) = 1$, it induces an algebra morphism $\tilde{f}\colon B[z^{-1}] \to E(\theta)$ such that the diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\;f\;} & E(\theta) \\
{\scriptstyle\iota}\downarrow & \nearrow{\scriptstyle\tilde f} & \\
B[z^{-1}] & &
\end{array}
$$

commutes, where ι is the inclusion map. Note that the algebra $B[z^{-1}]$ is a $\mathbb{Z}$-graded algebra, and $E(\theta)$ is a $\mathbb{Z}_2$-graded algebra. The next result shows that the degree zero parts $B[z^{-1}]_0$ and $E(\theta)_0$ are isomorphic as algebras, which is motivated by [20, Lemma 5.1].

Proposition 5.3. *The algebra morphism $\tilde{f}$ induces an isomorphism of algebras $B[z^{-1}]_0 \cong E(\theta)_0$.*

Proof. Since z is of degree 2, it follows that $B[z^{-1}]_0 = \sum_{i\geq 0} B_{2i}z^{-i}$. Then we have $\tilde{f}(B[z^{-1}]_0) = \sum_{i\geq 0} f(B_{2i}) = E(\theta)_0$ by the definition of $E(\theta)$ and the hypothesis $E = B/Bz$. Hence, $\tilde{f}$ induces an epimorphism $B[z^{-1}]_0 \to E(\theta)_0$. By Remark 4.2, $\dim E(\theta)_0 = \dim \left(\bigoplus_{i\geq 0} E_{2i}\right)$. By [20, Lemma 5.1(3)] and its proof, we have $\dim B[z^{-1}]_0 = \dim \left(\bigoplus_{i\geq 0} E_{2i}\right) = \dim E(\theta)_0$. Hence, the restriction of $\tilde{f}$ to $B[z^{-1}]_0$ yields an isomorphism of algebras $B[z^{-1}]_0 \cong E(\theta)_0$. $\square$

The structure of $E(\theta)_0$ can be easily determined if the Koszul Frobenius algebra E has lower dimension (see Section 10 for detailed computations of $E(\theta)_0$).

6 Noncommutative Quadric Hypersurfaces

Let A be a noetherian Artin-Schelter Gorenstein algebra, and let M be a right graded A-module. An element $m \in M$ is called a torsion element if $mA_{\geq n} = 0$ for some $n \geq 0$. Let $\Gamma(M)$ be the submodule of M consisting of all the torsion elements. Since A is noetherian, we get a functor $\Gamma\colon \operatorname{gr} A \to \operatorname{gr} A$. It is easy to see $\Gamma \cong \varinjlim \underline{\operatorname{Hom}}_A(A/A_{\geq n}, -)$. The ith right derived functor of Γ is written as $R^i\Gamma$.

For a finitely generated right graded A-module M, the *depth* of M is defined to be $\operatorname{depth}(M) = \min\{i \mid R^i\Gamma(M) \neq 0\}$. Assume that $\operatorname{injdim}{}_AA = \operatorname{injdim} A_A = d$. Then M is called a *maximal Cohen-Macaulay* (or *Gorenstein projective*) module if $\operatorname{depth}(M) = d$. Let $\operatorname{mcm} A$ be the subcategory of $\operatorname{gr} A$ consisting of all the maximal Cohen-Macaulay modules. The additive category $\operatorname{mcm} A$ is a Frobenius category with enough projectives and injectives. Let $\underline{\operatorname{mcm}}\, A$ be the stable category of $\operatorname{mcm} A$. Then $\underline{\operatorname{mcm}}\, A$ is a triangulated category.

Now let $S = T(V)/(R)$ be a Koszul Artin-Schelter regular algebra. Let $z \in S_2$ be a central regular element of S. The quotient algebra $A = S/Sz$ is usually called a (noncommutative) *quadric hypersurface*.

Let $\pi_S\colon T(V) \to S$ be the natural projection map. Pick an element $r_0 \in V \otimes V$ such that $\pi_S(r_0) = z$. Denote the quadratic dual algebra of S by $E = T(V^*)/(R^\perp)$. Then E is a Koszul Frobenius algebra (see Lemma 2.3). Since $A = S/Sz$, it follows that $A \cong T(V)/(\Bbbk r_0 + R)$. As before, write R' for the space $\Bbbk r_0 + R$. Note that $\Bbbk r_0 \cap R = 0$. Then we have $V \otimes V = \Bbbk r_0 \oplus R \oplus R''$ for some subspace $R'' \subseteq V \otimes V$. Define a linear map

$$r_0^*:\ V\otimes V\longrightarrow \Bbbk$$

by $r_0^*(r_0)=1$ and $r_0^*(R)=r_0^*(R'')=0$. We view r_0^* as an element of $V^*\otimes V^*$.

Consider the quadratic dual algebra $A^!=T(V^*)/(R'^{\perp})$. Note that we have $R'^{\perp}=(\Bbbk r_0)^{\perp}\cap R^{\perp}$. On the other hand, $R'^{\perp}+\Bbbk r_0^*=(\Bbbk r_0)^{\perp}\cap R^{\perp}+\Bbbk r_0^*=R^{\perp}$. Let $\pi_{A^!}\colon T(V^*)\to A^!$ be the projection map. Then $w:=\pi_{A^!}(r_0^*)\neq 0$. We next check that w is a central element of $A^!$. Note that $A^!_3\cong (R'\otimes V\cap V\otimes R')^*$ as vector spaces. For every $\alpha\in R'\otimes V\cap V\otimes R'$, by Lemma 5.1, $\alpha=v_0\otimes r_0+\sum_{i=1}^t v_i\otimes r_i=r_0\otimes v_0+\sum_{i=1}^t r_i\otimes v_i'$ for some $v_0,\dots,v_t,v_0',\dots,v_t'\in V$, where $r_1,\dots,r_t$ is a basis of R. Taking an element $f\in V^*\cong A^!_1$, we have $(fw)(\alpha)=f(v_0)r_0^*(r_0)=f(v_0)$ and $(wf)(\alpha)=r_0^*(r_0)f(v_0)=f(v_0)$. Hence, $fw=wf$ for all $f\in V^*$. Therefore, w is a central element. Now it follows that $E\cong A^!/A^!w$. Moreover, w is also a regular element of $A^!$ (see [20, Lemma 5.1(2)]).

For a $\mathbb{Z}_2$-graded algebra B, we write $\mathrm{Gr}_{\mathbb{Z}_2} B$ (resp., $\mathrm{gr}_{\mathbb{Z}_2} B$) for the category of right $\mathbb{Z}_2$-graded modules (resp., finitely generated $\mathbb{Z}_2$-graded modules), whose morphisms are degree 0 right $\mathbb{Z}_2$-graded morphisms. We arrive at our main observation of the paper, which is an improvement of [20, Proposition 5.2].

Theorem 6.1. *Let $S=T(V)/(R)$ be a noetherian Koszul Artin-Schelter regular algebra and let $E:=T(V^*)/(R^{\perp})$ be the quadratic dual algebra of S. Assume that $z\in S_2$ is a central regular element of S. Let $\theta_z\colon R^{\perp}\to\Bbbk$ be the map as defined in* (2)*. We have the following statements:*

- (i) *θ_z is a Clifford map of E, and hence $E(\theta_z)$ is a Clifford deformation of E.*
- (ii) [20, Lemma 5.1(1)] *Let $A=S/Sz$. Then A is a Koszul Artin-Schelter Gorenstien algebra.*
- (iii) *There is an equivalence of triangulated categories $\underline{\mathrm{mcm}}\,A\cong D^{\mathrm{b}}(\mathrm{gr}_{\mathbb{Z}_2} E(\theta_z))$.*

Proof. (i) follows from Lemma 3.9.

(ii) By Lemma 2.2, A is Artin-Schelter Gorenstein. Since S is a Koszul algebra and $z\in S_2$ is a central regular element, A is a Koszul algebra (see [20, Lemma 5.1(1)]).

(iii) Retain the notation as above. The element $w=\pi_{A^!}(r_0^*)$ is a central regular element of $A^!$. Let $A^![w^{-1}]$ be the localization of $A^!$ at w. Then $A^![w^{-1}]$ is a $\mathbb{Z}$-graded algebra. By [20, Proposition 5.2] there exists an equivalence of triangulated categories $D^{\mathrm{b}}(\mathrm{mod}\,A^![w^{-1}]_0)\cong\underline{\mathrm{mcm}}\,A$. Note that we have $R'^{\perp}=(\Bbbk r_0)^{\perp}\cap R^{\perp}$ and $R^{\perp}=R'^{\perp}+\Bbbk r_0^*$. Then $\theta_z(R'^{\perp})=0$ and $\theta_z(r_0^*)=1$. By Proposition 5.3,

$$A^![w^{-1}]_0\cong E(\theta_z)_0. \tag{7}$$

So $D^{\mathrm{b}}(\mathrm{mod}\,A^![w^{-1}]_0)\cong D^{\mathrm{b}}(\mathrm{mod}\,E(\theta_z)_0)$. By Proposition 4.6, $E(\theta_z)$ is a strongly $\mathbb{Z}_2$-graded algebra. It follows that there is an equivalence of abelian categories $\mathrm{gr}_{\mathbb{Z}_2} E(\theta_z)\cong\mathrm{mod}\,E(\theta_z)_0$ (see [13, Theorem 3.1.1]). Thus, $\underline{\mathrm{mcm}}\,A\cong D^{\mathrm{b}}(\mathrm{gr}_{\mathbb{Z}_2} E(\theta_z))$ as triangulated categories. □

7 Noncommutative Quadrics with Isolated Singularities

Let A be a noetherian connected graded algebra. Let $\mathrm{tor}\,A$ be the full subcategory of $\mathrm{gr}\,A$ consisting of finite dimensional right graded A-modules. The quotient category

$\operatorname{qgr} A = \operatorname{gr} A/\operatorname{tor} A$ is the noncommutative analogue of projective schemes (see [2, 22]). For $M \in \operatorname{gr} A$, we write $\mathcal{M}$ for the corresponding object in $\operatorname{qgr} A$. Recall from [21] that A is called a *noncommutative isolated singularity* if $\operatorname{qgr} A$ has finite global dimension, that is, there is an integer p such that for any objects $\mathcal{M}$, $\mathcal{N}$ in $\operatorname{qgr} A$, $\operatorname{Ext}^i_{\operatorname{qgr} A}(\mathcal{M}, \mathcal{N}) = 0$ for all $i > p$, or equivalently, the noncommutative projective scheme $\operatorname{Proj} A$ is *smooth* (see [20]).

Let S be a noetherian Koszul Artin-Schelter regular algebra. Assume that $z \in S_2$ is a central regular element of S. In this section we investigate when $A = S/Sz$ is a noncommutative isolated singularity.

Let $\operatorname{per} A$ be the triangulated subcategory of $D^{\mathrm{b}}(\operatorname{gr} A)$ consisting of bounded complexes of finitely generated right graded projective A-modules. Then we have a quotient triangulated category $D^{\mathrm{gr}}_{\mathrm{sg}}(A) = D^{\mathrm{b}}(\operatorname{gr} A)/\operatorname{per} A$.

Since A is Artin-Schelter Gorenstein, there is an equivalence of triangulated categories [3, Theorem 4.4.1(2)]:

$$D^{\mathrm{gr}}_{\mathrm{sg}}(A) \cong \underline{\mathrm{mcm}}\, A. \tag{8}$$

The triangulated category $D^{\mathrm{gr}}_{\mathrm{sg}}(A)$ is related to $D^{\mathrm{b}}(\operatorname{qgr} A)$ by Orlov's famous decomposition theorem.

Theorem 7.1. [14, Theorem 2.5] *Let A be a noetherian Artin-Schelter Gorenstein algebra of Gorenstein parameter l. Then the following statements hold:*

(i) *If $l > 0$, then there are fully faithful functors $\Phi_i \colon D^{\mathrm{gr}}_{\mathrm{sg}}(A) \to D^{\mathrm{b}}(\operatorname{qgr} A)$ and semiorthogonal decompositions*

$$D^{\mathrm{b}}(\operatorname{qgr} A) = \langle \pi A(-i-l+1), \ldots, \pi A(-i), \Phi_i(D^{\mathrm{gr}}_{\mathrm{sg}}(A))\rangle,$$

where $\pi \colon \operatorname{gr} A \to \operatorname{qgr} A$ is the projection functor.

(ii) *If $l < 0$, then there are fully faithful functors $\Psi_i \colon D^{\mathrm{b}}(\operatorname{qgr} A) \to D^{\mathrm{gr}}_{\mathrm{sg}}(A)$ and semiorthogonal decompositions*

$$D^{\mathrm{gr}}_{\mathrm{sg}}(A) = \langle q\Bbbk(-i), \ldots, q\Bbbk(-i+l+1), \Psi_i(D^{\mathrm{b}}(\operatorname{qgr} A))\rangle,$$

where $q \colon D^{\mathrm{b}}(\operatorname{gr} A) \to D^{\mathrm{gr}}_{\mathrm{sg}}(A)$ is the natural projection functor.

(iii) *If $l = 0$, then there is an equivalence $D^{\mathrm{gr}}_{\mathrm{sg}}(A) \cong D^{\mathrm{b}}(\operatorname{qgr} A)$.*

We have the following special case of Orlov's theorem.

Lemma 7.2. *Let S be a noetherian Koszul Artin-Schelter regular algebra of global dimension $d \geq 2$, and let $z \in S_2$ be a central regular element of S. Set $A = S/Sz$. Then there is a fully faithful triangle functor $\Phi \colon \underline{\mathrm{mcm}}\, A \to D^{\mathrm{b}}(\operatorname{qgr} A)$.*

Proof. Since S is a Koszul Artin-Schelter regular algebra of global dimension d, the Gorenstein parameter of S is equal to d. By Lemma 2.2, A is Artin-Schelter Gorenstein of injective dimension $d-1$ with Gorenstein parameter $d-2 \geq 0$. The lemma follows from the equivalence (8) and Theorem 7.1(i)(iii). □

Now let $S = T(V)/(R)$ be a noetherian Koszul Artin-Schelter regular algebra of global dimension $d \geq 2$, and $E = T(V^*)/(R^{\perp})$ its quadratic dual. Assume that

$z \in S_2$ is a central regular element of S. Let $E(\theta_z)$ be the Clifford deformation of E corresponding to the central element z (cf. Theorem 6.1).

Theorem 7.3. *Retain the notation as above. Then $A = S/Sz$ is a noncommutative isolated singularity if and only if $E(\theta_z)$ is semisimple as a $\mathbb{Z}_2$-graded algebra.*

Proof. Assume that $E(\theta_z)$ is a $\mathbb{Z}_2$-graded semisimple algebra. By Proposition 4.6, $E(\theta_z)$ is a strongly $\mathbb{Z}_2$-graded algebra. Then $\operatorname{gr}_{\mathbb{Z}_2} E(\theta_z) \cong \operatorname{mod} E(\theta_z)_0$ as abelian categories. Hence, $E(\theta_z)_0$ is a semisimple algebra. By the isomorphism (7) in the proof of Theorem 6.1, $E(\theta_z)_0 \cong A^![w^{-1}]_0$. Thus, $A^![w^{-1}]_0$ is semisimple. By [20, Proposition 5.2(2)], A is a noncommutative isolated singularity.

Conversely, assume that A is a noncommutative isolated singularity. Then $\operatorname{qgr} A$ has finite global dimension. Given objects $X, Y \in D^{\mathrm{b}}(\operatorname{qgr} A)$, there is an integer p (depending on X and Y) such that $\operatorname{Hom}_{D^{\mathrm{b}}(\operatorname{qgr} A)}(X, Y[i]) = 0$ for $i > p$. Let J be the $\mathbb{Z}_2$-graded Jacobson radical of $E(\theta_z)$. Write T for the quotient algebra $E(\theta_z)/J$. By Theorem 6.1, there exists an equivalence of triangulated categories $\Psi\colon D^{\mathrm{b}}(\operatorname{gr}_{\mathbb{Z}_2} E(\theta_z)) \to \underline{\operatorname{mcm}}\, A$. Let $\Phi\colon \underline{\operatorname{mcm}}\, A \to D^{\mathrm{b}}(\operatorname{qgr} A)$ be the fully faithful functor in Lemma 7.2. Then there is an integer q such that for $i > q$,

$$\begin{aligned}\operatorname{Ext}^i_{\operatorname{gr}_{\mathbb{Z}_2} E(\theta_z)}(T, T) &\cong \operatorname{Hom}_{D^{\mathrm{b}}(\operatorname{gr}_{\mathbb{Z}_2} E(\theta_z))}(T, T[i]) \\ &\cong \operatorname{Hom}_{D^{\mathrm{b}}(\operatorname{qgr} A)}(\Phi\Psi(T), \Phi\Psi(T)[i]) = 0.\end{aligned}$$

Since $E(\theta_z)$ is finite dimensional, as a right $\mathbb{Z}_2$-graded $E(\theta_z)$-module T is semisimple and each simple right $\mathbb{Z}_2$-graded $E(\theta_z)$-module is a direct summand of T. It follows that the right $\mathbb{Z}_2$-graded $E(\theta_z)$-module T has finite projective dimension in $\operatorname{gr}_{\mathbb{Z}_2} E(\theta_z)$. Hence, the $\mathbb{Z}_2$-graded algebra $E(\theta_z)$ has finite global dimension. On the other hand, $E(\theta_z)$ is a $\mathbb{Z}_2$-graded Frobenius algebra by Proposition 4.5. It follows that $E(\theta_z)$ is semisimple as a $\mathbb{Z}_2$-graded algebra. □

Remark 7.4. (i) The sufficiency part of Theorem 7.3 mainly follows from [20, Proposition 5.2]. Observing that $E(\theta_z)$ is strongly $\mathbb{Z}_2$-graded, we obtain an abstract proof of [20, Theorem 5.6].

(ii) Independently, Mori and Ueyama gave another proof of the above theorem in [12, Theorem 5.4].

8 Clifford Deformations of Trivial Extensions of Koszul Frobenius Algebras

In this section, we work over the field of complex numbers $\mathbb{C}$. Let $\mathbb{M}_2(\mathbb{C})$ be the matrix algebra of all the 2×2-matrices over $\mathbb{C}$. We may view $\mathbb{M}_2(\mathbb{C})$ as a $\mathbb{Z}_2$-graded algebra by setting the degree 0 part to consist of elements of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, and degree 1 part to consist of elements of the form $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$. Let A and B be $\mathbb{Z}_2$-graded algebras. The twisting tensor algebra $A \hat{\otimes} B$ is a $\mathbb{Z}_2$-graded algebra defined as follows: $(A \hat{\otimes} B)_0 = A_0 \otimes B_0 \oplus A_1 \otimes B_1$, $(A \hat{\otimes} B)_1 = A_0 \otimes B_1 \oplus A_1 \otimes B_0$ as a $\mathbb{Z}_2$-graded space; the multiplication is defined as $(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{|b_1||a_2|}(a_1a_2 \hat{\otimes} b_1b_2)$, where b_1 and a_2 are homogeneous elements with degrees $|b_1|$ and $|a_2|$, respectively.

Let $\mathbb{G} = \{1, \alpha\}$ be a group of order 2, and let $\mathbb{C}\mathbb{G}$ be the group algebra. Then $\mathbb{C}\mathbb{G}$ is naturally a $\mathbb{Z}_2$-graded algebra by setting $|\alpha| = 1$ and $|1| = 0$. Define a linear map

$$\Upsilon : \ \mathbb{C}\mathbb{G} \hat{\otimes} \mathbb{C}\mathbb{G} \longrightarrow \mathbb{M}_2(\mathbb{C})$$

by $1 \hat{\otimes} 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $1 \hat{\otimes} \alpha \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\alpha \hat{\otimes} 1 \mapsto \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$, $\alpha \hat{\otimes} \alpha \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$. The following lemma is well known and is easy to check.

Lemma 8.1. *The map Υ is an isomorphism of $\mathbb{Z}_2$-graded algebras.*

Note that there is an equivalence of abelian categories $\operatorname{gr}_{\mathbb{Z}_2} \mathbb{M}_2(\mathbb{C}) \cong \operatorname{gr}_{\mathbb{Z}_2} \mathbb{C}$, where $\mathbb{C}$ is viewed as a $\mathbb{Z}_2$-graded algebra concentrated in degree 0. By the above lemma we have an equivalence of abelian categories (see also [24, Lemma 4.11]) $\operatorname{gr}_{\mathbb{Z}_2}(\mathbb{C}\mathbb{G} \hat{\otimes} \mathbb{C}\mathbb{G}) \cong \operatorname{gr}_{\mathbb{Z}_2} \mathbb{C}$. We have the following result, which should be well known for experts.

Lemma 8.2. *Let A be a $\mathbb{Z}_2$-graded algebra. Then there is an equivalence of abelian categories $\operatorname{gr}_{\mathbb{Z}_2}(A \hat{\otimes} \mathbb{C}\mathbb{G} \hat{\otimes} \mathbb{C}\mathbb{G}) \cong \operatorname{gr}_{\mathbb{Z}_2} A$.*

Proof. This is a direct consequence of [24, Lemma 3.10]. Note that the algebra is assumed to be finite dimensional in [24, Lemma 3.10], but the result hold for arbitrary $\mathbb{Z}_2$-graded algebras. □

Lemma 8.3. *For $P \in \operatorname{gr}_{\mathbb{Z}_2} A \hat{\otimes} \mathbb{C}\mathbb{G}$, P is projective if and only if P is projective as a $\mathbb{Z}_2$-graded A-module.*

Proof. Note that $A \hat{\otimes} \mathbb{C}\mathbb{G}$ is a $\mathbb{Z}_2$-graded projective A-module, and hence each $\mathbb{Z}_2$-graded projective $A \hat{\otimes} \mathbb{C}\mathbb{G}$-module is projective as a $\mathbb{Z}_2$-graded A-module.

Conversely, we suppose that P is projective as a $\mathbb{Z}_2$-graded A-module. Let $f \colon M \to N$ be an epimorphism of $\mathbb{Z}_2$-graded $A \hat{\otimes} \mathbb{C}\mathbb{G}$-modules and $g \colon P \to N$ be a $\mathbb{Z}_2$-graded $A \hat{\otimes} \mathbb{C}\mathbb{G}$-module morphism. Since P is projective as a graded A-module, there is a graded A-module morphism $h \colon P \to M$ such that $fh = g$. Define a morphism $h' \colon P \to M$ by $h'(p) = \frac{1}{2}(h(p) + h(p \cdot \alpha) \cdot \alpha)$ for all $p \in P$. It is straightforward to check that h' is a graded $A \hat{\otimes} \mathbb{C}\mathbb{G}$-module morphism, and that $fh' = g$. Hence, P is projective in $\operatorname{gr}_{\mathbb{Z}_2} A \hat{\otimes} \mathbb{C}\mathbb{G}$. □

Let $\operatorname{gldim}_{\mathbb{Z}_2} A$ denote the $\mathbb{Z}_2$-graded global dimension of A.

Corollary 8.4. $\operatorname{gldim}_{\mathbb{Z}_2} A = \operatorname{gldim}_{\mathbb{Z}_2} A \hat{\otimes} \mathbb{C}\mathbb{G}$.

Proof. Let M be a right $\mathbb{Z}_2$-graded $A \hat{\otimes} \mathbb{C}\mathbb{G}$-module. Assume $\operatorname{gldim}_{\mathbb{Z}_2} A = d < \infty$. Suppose that

$$\cdots \longrightarrow P_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

is a projective resolution of M in $\operatorname{gr}_{\mathbb{Z}_2} A \hat{\otimes} \mathbb{C}\mathbb{G}$. Since $\operatorname{gldim}_{\mathbb{Z}_2} A = d$, we see that $Q = \ker \delta_{d-1}$ is projective as a graded A-module, and hence Q is a projective graded $A \hat{\otimes} \mathbb{C}\mathbb{G}$-module by Lemma 8.3. Therefore, $\operatorname{gldim}_{\mathbb{Z}_2} A \hat{\otimes} \mathbb{C}\mathbb{G} \leq \operatorname{gldim}_{\mathbb{Z}_2} A$. Similarly, we obtain $\operatorname{gldim}_{\mathbb{Z}_2} A \hat{\otimes} \mathbb{C}\mathbb{G} \hat{\otimes} \mathbb{C}\mathbb{G} \leq \operatorname{gldim}_{\mathbb{Z}_2} A \hat{\otimes} \mathbb{C}\mathbb{G}$. By Lemma 8.2, it follows that

$\operatorname{gldim}_{\mathbb{Z}_2} A = \operatorname{gldim}_{\mathbb{Z}_2} A \hat{\otimes} \mathbb{CG} \hat{\otimes} \mathbb{CG}$.

If $\operatorname{gldim}_{\mathbb{Z}_2} A = \infty$, we claim $\operatorname{gldim}_{\mathbb{Z}_2} A \hat{\otimes} \mathbb{CG} = \infty$. Indeed, if $\operatorname{gldim}_{\mathbb{Z}_2} A \hat{\otimes} \mathbb{CG} < \infty$, by the above proof it follows that $\operatorname{gldim}_{\mathbb{Z}_2} A \hat{\otimes} \mathbb{CG} \hat{\otimes} \mathbb{CG} \leq \operatorname{gldim}_{\mathbb{Z}_2} A \hat{\otimes} \mathbb{CG}$. Thus, by Lemma 8.2, $\operatorname{gldim}_{\mathbb{Z}_2} A < \infty$, a contradiction. Hence, the result follows. □

Let $E = T(V)/(R)$ be a Koszul Frobenius algebra, and M a graded E-bimodule. The trivial extension of E by M is the $\mathbb{Z}$-graded algebra $\Gamma(E, M) = E \oplus M$ with a multiplication defined by $(x_1, m)(x_2, n) = (x_1x_2,\ x_1n + mx_2)$ for $x_1, x_2 \in E$ and $m, n \in M$. Let ϵ be the automorphism of E defined by $\epsilon(x) = (-1)^{|x|}x$ for homogeneous elements $x \in E$. Let ${}_{\epsilon}E$ be the graded E-bimodule with left E-action twisted by ϵ. Define

$$\tilde{E} = \Gamma(E,\ {}_{\epsilon}E(-1)). \tag{9}$$

Lemma 8.5. *Let E be a Koszul Frobenius algebra. Then $\tilde{E}$ is a Koszul Frobenius algebra.*

Proof. Let $S = E^! = T(V^*)/(R^{\perp})$ be the quadratic dual of E. Then S is a Koszul Artin-Schelter regular algebra by Lemma 2.3. Let $S^{\natural} = S[\alpha]$ be the polynomial algebra with coefficients in S. It is well known that $S^{\natural}$ is a Koszul Artin-Schelter regular algebra. Let $V^{\natural} = V^* \oplus \Bbbk\alpha$ and $R^{\natural} = R^{\perp} \oplus \operatorname{span}\{\beta \otimes \alpha - \alpha \otimes \beta \mid \beta \in V^*\}$. The Koszul algebra $S^{\natural}$ may be written as $S^{\natural} = T(V^{\natural})/(R^{\natural})$.

By [6, Proposition 2.2], it follows that $\tilde{E}$ is the quadratic dual algebra of $S^{\natural}$. By Lemma 2.3, we know that $\tilde{E}$ is Koszul Frobenius. □

We may write the trivial extension $\tilde{E}$ by generators and relations. Assume that $\tilde{V} = V \oplus \Bbbk y$ and let $\tilde{R} = R \oplus R' \oplus \Bbbk y \otimes y$, where $R' = \operatorname{span}\{x \otimes y + y \otimes x \,|\, x \in V\}$. Then $\tilde{E} = T(\tilde{V})/(\tilde{R})$.

Assume that $\theta\colon R \to \Bbbk$ is a Clifford map of E. Define a linear map

$$\tilde{\theta}:\ \tilde{R} \longrightarrow \Bbbk \quad \text{by} \tag{10}$$

$$\tilde{\theta}(r) = \theta(r) \text{ for } r \in R,\quad \tilde{\theta}(y \otimes y) = 1,\quad \tilde{\theta}(x \otimes y + y \otimes x) = 0 \text{ for } x \in V. \tag{11}$$

Lemma 8.6. *Retain the notation as above. Then $\tilde{\theta}$ is a Clifford map of the Koszul Frobenius algebra $\tilde{E}$, and hence $\tilde{E}(\tilde{\theta})$ is a Clifford deformation of $\tilde{E}$.*

Proof. Let $\{x_1, \dots, x_n\}$ be a basis of V. Then $\{x_i \otimes y + y \otimes x_i \,|\, i = 1, \dots, n\}$ is a basis of R'. We may write

$$\tilde{V} \otimes \tilde{R} = (V \otimes R) \oplus (V \otimes R') \oplus (V \otimes y \otimes y) \oplus (y \otimes R) \oplus (y \otimes R') \oplus (\Bbbk y \otimes y \otimes y),$$

and similarly,

$$\tilde{R} \otimes \tilde{V} = (R \otimes V) \oplus (R' \otimes V) \oplus (y \otimes y \otimes V) \oplus (R \otimes y) \oplus (R' \otimes y) \oplus (\Bbbk y \otimes y \otimes y).$$

For an element $w \in \tilde{V} \otimes \tilde{R}$ we may write $w = w_1 + w_2 + w_3 + w_4 + w_5 + ky \otimes y \otimes y$ for $w_1 \in V \otimes R$, $w_2 \in V \otimes R'$, $w_3 \in V \otimes y \otimes y$, $w_4 \in y \otimes R$, and $w_5 \in y \otimes R'$. Now assume that w is also in $\tilde{R} \otimes \tilde{V}$. By comparing the multiplicity of the element z in the tensor products, we see that $w_1 \in R \otimes V$, $w_2 + w_4 \in R' \otimes V \oplus R \otimes y$, $w_3 + w_5 \in y \otimes y \otimes V \oplus R' \otimes y$. Assume $w_2 = \sum_{i,j=1}^{n} a_{ij} x_i \otimes (x_j \otimes y - y \otimes x_j)$ and $w_4 = y \otimes \sum_{i,j=1}^{n} b_{ij} x_i \otimes x_j$. Then

$$\begin{aligned} w_2+w_4 &= \sum_{i,j=1}^{n} a_{ij}x_i\otimes(x_j\otimes y-y\otimes x_j)+y\otimes\sum_{i,j=1}^{n} b_{ij}x_i\otimes x_j \\ &= \sum_{i,j=1}^{n} a_{ij}(x_i\otimes x_j)\otimes y+\sum_{j=1}^{n}\sum_{i=1}^{n}(-a_{ij}x_i\otimes y+b_{ij}y\otimes x_i)\otimes x_j. \end{aligned}$$

Since $w_2+w_4\in R'\otimes V\oplus R\otimes z$, we have $\sum_{j=1}^{n}\sum_{i=1}^{n}(-a_{ij}x_i\otimes y+b_{ij}y\otimes x_i)\otimes x_j\in R'\otimes V$. Then $a_{ij}=b_{ij}$ for all i and j, and it follows that

$$(1\otimes\tilde\theta)(w_2+w_4)=\theta\Big(\sum_{i,j=1}^{n} a_{ij}x_i\otimes x_j\Big)y=(\tilde\theta\otimes 1)(w_2+w_4).$$

Similarly, we have $(1\otimes\tilde\theta)(w_3+w_5)=(\tilde\theta\otimes 1)(w_3+w_5)$. Since θ is a Clifford map of E, we get $(1\otimes\theta)(w_1)=(\theta\otimes 1)(w_1)$. Therefore, $(1\otimes\tilde\theta)(w)=(\tilde\theta\otimes 1)(w)$. □

Proposition 8.7. *Retain the notation as above. There is an isomorphism of $\mathbb{Z}_2$-graded algebras $\tilde E(\tilde\theta)\cong E(\theta)\,\hat\otimes\,\mathbb{C}\mathbb{G}$.*

Proof. Note that $E(\theta)=T(V)/(r-\theta(r): r\in R)$, $\tilde E(\tilde\theta)=T(\tilde V)/(r-\tilde\theta(r): r\in\tilde R)$. Let $\pi\colon T(V)\to E(\theta)$ be the projection map. Now we may define a linear map $\psi\colon T(\tilde V)\to E(\theta)\,\hat\otimes\,\mathbb{C}\mathbb{G}$ by setting $\psi(x)=\pi(x)\,\hat\otimes\,1$ for $x\in V$ and $\psi(y)=1\,\hat\otimes\,\alpha$. The map ψ induces an algebra epimorphism $\overline{\psi}\colon\tilde E(\tilde\theta)\to E(\theta)\,\hat\otimes\,\mathbb{C}\mathbb{G}$. Clearly, $\overline{\psi}$ preserves the $\mathbb{Z}_2$-grading. Since $E(\theta)\,\hat\otimes\,\mathbb{C}\mathbb{G}$ and $\tilde E(\tilde\theta)$ have the same dimension as vector spaces, $\overline{\psi}$ is an isomorphism. □

As a special case of Corollary 8.4, we have the following result.

Corollary 8.8. $\operatorname{gldim}_{\mathbb{Z}_2}\tilde E(\tilde\theta)=\operatorname{gldim}_{\mathbb{Z}_2}E(\theta)$.

We may iterate the above construction. By Lemma 8.5, $\tilde E$ is a Koszul Frobenius algebra, and the trivial extension $\tilde{\tilde E}$ of $\tilde E$ is again a Koszul Frobenius algebra. The Clifford map $\tilde\theta$ of the Koszul algebra $\tilde E$ may be extended to a Clifford map $\tilde{\tilde\theta}$ of $\tilde{\tilde E}$ in the same way as in (10) and (11). Hence, we obtain a $\mathbb{Z}_2$-graded algebra $\tilde{\tilde E}(\tilde{\tilde\theta})$.

Proposition 8.9. *There is an equivalence $\operatorname{gr}_{\mathbb{Z}_2}\tilde{\tilde E}(\tilde{\tilde\theta})\cong\operatorname{gr}_{\mathbb{Z}_2}E(\theta)$ of abelian categories.*

Proof. By Proposition 8.7, $\tilde{\tilde E}(\tilde{\tilde\theta})\cong\tilde E(\tilde\theta)\,\hat\otimes\,\mathbb{C}\mathbb{G}\cong E(\theta)\,\hat\otimes\,\mathbb{C}\mathbb{G}\,\hat\otimes\,\mathbb{C}\mathbb{G}$. Now the result follows from Lemma 8.2. □

9 Knörrer's Periodicity for Noncommutative Quadrics Revisited

In this section, the base field is assumed to be $\Bbbk=\mathbb{C}$. Let $S=T(V)/(R)$ be a noetherian Koszul Artin-Schelter regular algebra of $\operatorname{gldim}A\geq 2$. Let $z\in S_2$ be a central regular element of S, and set $A=S/(z)$. The *double branched cover* of A is defined to be the algebra (cf. [4, 7, 8])

$$A^{\#}:=S[v_1]/(z+v_1^2).$$

We write $S[v_1] = T(U)/(R')$ with $U = V \oplus \Bbbk v_1$ and $R' = R \oplus \{v \otimes v_1 - v_1 \otimes v \mid v \in V\}$.

Let $E = S^!$ be the quadratic dual algebra of S, and $\tilde{E}$ the trivial extension of E as defined by (9). Then by [6, Proposition 2.2], $\tilde{E}$ is isomorphic to the quadratic dual algebra $S[v_1]^!$.

Denote by $\pi_S \colon T(V) \to S$ the natural projection map. Pick an element r_0 in $V \otimes V$ such that $\pi_S(r_0) = z$. Let θ_z be the Clifford map as defined by (2) (see Lemma 3.9), and $E(\theta_z)$ the Clifford deformation of E associated to θ_z.

Let $\pi_{S[v_1]} \colon T(U) \to S[v_1]$ be the projection map. Set $r_0^{\#} = r_0 + v_1 \otimes v_1$. Then $\pi_{S[v_1]}(r_0^{\#}) = z + v_1^2$. Let $\tilde{\theta}_z$ be the composition $R'^{\perp} \hookrightarrow U^* \otimes U^* \xrightarrow{r_0^{\#}} \Bbbk$. Since $z + v_1^2$ is a central element, $\tilde{\theta}_z$ is a Clifford map of $\tilde{E}$.

The main purpose of this section is to recover Knörrer's periodicity theorem in the case of quadric hypersurface singularities. Firstly, we recover [7, Corollary 2.8] and [4, Theorem 1.6] for quadrics without using matrix factorizations.

Theorem 9.1. *Retain the notation as above. Then the algebra A is a noncommutative isolated singularity if and only if so is $A^{\#}$.*

Proof. By Theorem 7.3, A (resp., $A^{\#}$) is a noncommutative isolated singularity if and only if $E(\theta_z)$ (resp., $\tilde{E}(\tilde{\theta}_z)$) is a $\mathbb{Z}_2$-graded semisimple algebra. By Corollary 8.8, $E(\theta_z)$ is a $\mathbb{Z}_2$-graded semisimple algebra if and only if so is $\tilde{E}(\tilde{\theta}_z)$. □

The *second double branched cover* of A is defined to be the algebra

$$A^{\#\#} := S[v_1, v_2]/(z + v_1^2 + v_2^2).$$

Let $W = V \oplus \Bbbk v_1 \oplus \Bbbk v_2$. Then $S[v_1, v_2] = T(W)/(R'')$, where

$$R'' = R \oplus \{v \otimes v_1 - v_1 \otimes v \mid v \in V\} \oplus \{v \otimes v_2 - v_2 \otimes v \mid v \in V\} \oplus \mathbb{C}(v_1 \otimes v_2 - v_2 \otimes v_1).$$

Let $\pi_{S[v_1,v_2]} \colon T(W) \to S[v_1, v_2]$ be the projection map. Furthermore, define $r_0^{\#\#} = r_0 + v_1 \otimes v_1 + v_2 \otimes v_2$. Then $\pi_{S[v_1,v_2]}(r_0^{\#\#}) = z + v_1^2 + v_2^2$. Let $\tilde{\tilde{\theta}}_z$ be the composition $R''^{\perp} \hookrightarrow W^* \otimes W^* \xrightarrow{r_0^{\#\#}} \Bbbk$. Then $\tilde{\tilde{\theta}}_z$ is a Clifford map associated to $\tilde{\tilde{E}}$, which is corresponding to the map obtained in (11).

We recover Knörrer's periodicity theorem (cf. [7, Theorem 3.1] and [4, Theorem 1.7]) for quadrics as follows, without using matrix factorizations.

Theorem 9.2. *Retain the notation as above. Then there is an equivalence of triangulated categories $\underline{\mathrm{mcm}}\, A \cong \underline{\mathrm{mcm}}\, A^{\#\#}$.*

Proof. We have equivalences of triangulated categories $\underline{\mathrm{mcm}}\, A \cong D^{\mathrm{b}}(\mathrm{gr}_{\mathbb{Z}_2} E(\theta_z))$ and $\underline{\mathrm{mcm}}\, A^{\#\#} \cong D^{\mathrm{b}}(\mathrm{gr}_{\mathbb{Z}_2} \tilde{\tilde{E}}(\tilde{\tilde{\theta}}_z))$ by Theorem 6.1(iii). By Proposition 8.9, we get an equivalence of abelian categories $\mathrm{gr}_{\mathbb{Z}_2} E(\theta_z) \cong \mathrm{gr}_{\mathbb{Z}_2} \tilde{\tilde{E}}(\tilde{\tilde{\theta}}_z)$. The result follows. □

10 Examples

In this section, we will list all the possible noncommutative quadric hypersurfaces obtained from an Artin-Schelter regular algebra of type S_2 as listed in [1, Table

3.11]. The key point in computation is the analysis of structures of degree zero part of Clifford deformations.

Let U be a 3-dimensional vector space with a fixed basis X, Y, Z, and in this section we assume $S = \Bbbk\langle X, Y, Z\rangle/(f_1, f_2, f_3)$, where

$$f_1 = ZX + XZ, \quad f_2 = YZ + ZY, \quad f_3 = X^2 + Y^2.$$

The associated algebra S is indeed the Artin-Schelter regular algebra of type S_2 as listed in [1, Table 3.11]. Thus, S is a Koszul algebra of global dimension 3, which is also a domain.

Let $V = U^*$ be the dual space. To simplify the notation, we write x, y, z for the basis of V dual to X, Y, Z. The quadratic dual $E\,(= S^!)$ of S has been computed in Example 3.6, which is the algebra $E = T(V)/(R)$, where R is the subspace of $V \otimes V$ spanned by $xz - zx$, $yz - zy$, $x^2 - y^2$, z^2, xy, yx. The basis of $V \otimes R \cap R \otimes V$ has been listed in Example 3.6. Now it is easy to check that the only possible nontrivial Clifford maps are defined as follows:

$$\theta\colon R \longrightarrow \Bbbk, \quad \theta(z^2) = \alpha,\ \theta(xy) = \theta(yx) = \beta,\ \theta(x^2 - y^2) = \lambda,\ \theta(\text{others}) = 0, \quad (12)$$

where $0 \neq (\alpha, \beta, \lambda) \in \Bbbk^3$.

Let $C = E(\theta)_0$. Then C is a commutative algebra and has a basis $1, xz, yz, x^2$. Write $a = xz$, $b = yz$, $c = x^2$. We have the following table of multiplications of C:

Table 1

	1	a	b	c
1	1	a	b	c
a	a	αc	$\alpha\beta$	$\lambda a + \beta b$
b	b	$\alpha\beta$	$\alpha c - \lambda\alpha$	βa
c	c	$\lambda a + \beta b$	$\beta\alpha$	$\lambda c + \beta^2$

We make a detailed analysis of the structures of C by choosing different scalars (α, β, λ). Since C is 4-dimensional, either C is semisimple or C/J is isomorphic to a product of $\Bbbk$, where J is the Jacobson radical of C.

We may assume that $\alpha = 1$ and $\beta = 1$ (one may replace z by $\sqrt{\alpha}z$, x by $\sqrt{\beta}x$, and y by $\sqrt{\beta}y$, if necessary).

Case (i). $\alpha \neq 0$, $\beta \neq 0$.

By Table 1, the commutative algebra C has the relations $a^2 = c$, $ab = 1$, $ac = \lambda a + b$, $b^2 = c - \lambda$, $bc = a$, $c^2 = \lambda c + 1$. One sees that C is generated by a. By $c^2 = \lambda c + 1$, one has $C \cong \Bbbk[a]/(f)$, where $f = a^4 - \lambda a^2 - 1$. If $\lambda \neq \pm 2\sqrt{-1}$, then f does not have multiple roots. Thus, $C \cong \Bbbk[a]/(f) \cong \Bbbk^4$. If $\lambda = \pm 2\sqrt{-1}$, then $f = (a^2 \pm \sqrt{-1})^2$. It follows that $C \cong \Bbbk[u]/(u^2) \times \Bbbk[u]/(u^2)$.

Case (ii). $\alpha = 1$, $\beta = 0$, $\lambda = 0$.

The commutative algebra C has relations $a^2 = c$, $b^2 = c$, $ab = ac = bc = c^2 = 0$. Then $C \cong \Bbbk[u, v]/(uv, u^2 - v^2) \cong \Bbbk[u, v]/(u^2, v^2)$.

Case (iii). $\alpha = 1$, $\beta = 0$, $\lambda = 1$.

By Table 1, the commutative algebra C has relations $a^2 = c$, $ac = a$, $b^2 = c - 1$, $c^2 = c$, $ab = bc = 0$. Let e_1, e_2, e_3, e_4 be the set of complete primitive idempotents of $\Bbbk^4$. Then $C \cong \Bbbk^4$ by setting $a \mapsto e_1 - e_2$, $b \mapsto \sqrt{-1}(e_3 + e_4)$, $c \mapsto e_1 + e_2$.

Case (iv). $\alpha = 0$, $\beta = 1$.

By Table 1, C has the relations $a^2 = ab = b^2 = 0$, $ac = \lambda a + b$, $bc = a$, $c^2 = \lambda c + 1$. If $\lambda = \pm 2\sqrt{-1}$, then $C \cong \Bbbk[u,v]/(u^2, v^2)$. The isomorphism is defined by $u \mapsto a + \frac{\lambda}{2}b$, $v \mapsto c - \frac{\lambda}{2}$. Assume $\lambda \neq \pm 2\sqrt{-1}$. By the identity $c^2 - \lambda c - 1 = 0$, we have $(c - t_1)(c - t_2) = 0$, where $t_1 = \frac{\lambda + \sqrt{\lambda^2+1}}{2}$ and $t_2 = \frac{\lambda - \sqrt{\lambda^2+4}}{2}$. Let $p = \frac{t_1}{t_1 t_2 - t_1^2}$ and $q = \frac{t_2}{t_1 t_2 - t_2^2}$. Then $e_1 = p(c - t_1)$ and $e_2 = q(c - t_2)$ are idempotents such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Thus, $C \cong \Bbbk[u]/(u^2) \times \Bbbk[u]/(u^2)$. The isomorphism is defined by $(u, 0) \mapsto a + \frac{\lambda - \sqrt{\lambda^2+4}}{2}b$, $(0, u) \mapsto a + \frac{\lambda + \sqrt{\lambda^2+4}}{2}b$.

Case (v). $\alpha = \beta = 0$, $\lambda = 1$.

By Table 1, one has $a^2 = ab = bc = b^2 = 0$, $ac = a$ and $c^2 = c$. Then c and $1 - c$ are primitive idempotents. It follows that $C \cong \Bbbk[u]/(u^2) \times \Bbbk[u]/(u^2)$.

Summarizing, we have the following table of all the possible noncommutative quadric hypersurfaces defined by a central element $w \in S_2$ of S (recall that a Clifford map θ corresponds to a central element in S_2; see Lemma 3.8), in which (α, β, λ) is the scalar of the Clifford map as defined in (12).

Table 2

No.	(α,β,λ)	w	$C = E(\theta)_0$	singularities of S/wS
1	$(1, 1, \lambda \neq \pm 2\sqrt{-1})$	$Z^2 + XY + YX + \lambda X^2$	$\Bbbk^4$	isolated
2	$(1, 1, \pm 2\sqrt{-1})$	$Z^2 + XY + YX \pm 2\sqrt{-1}X^2$	$\Bbbk[u]/(u^2) \times \Bbbk[u]/(u^2)$	nonisolated
3	$(1, 0, 0)$	Z^2	$\Bbbk[u,v]/(u^2, v^2)$	nonisolated
4	$(1, 0, 1)$	$Z^2 + X^2$	$\Bbbk^4$	isolated
5	$(0, 1, \lambda \neq \pm 2\sqrt{-1})$	$XY + YX + \lambda X^2$	$\Bbbk[u]/(u^2) \times \Bbbk[u]/(u^2)$	nonisolated
6	$(0, 1, \pm 2\sqrt{-1})$	$XY + YX \pm 2\sqrt{-1}X^2$	$\Bbbk[u,v]/(u^2, v^2)$	nonisolated
7	$(0, 0, 1)$	X^2	$\Bbbk[u]/(u^2) \times \Bbbk[u]/(u^2)$	nonisolated

Note that the noncommutative quadrics Nos. 1 and 4 in Table 2 are isolated singularities. We remark that the associated Clifford deformations of these two cases are both isomorphic to $\mathbb{C}\mathbb{G}^{\times 4}$. This is because the associated Clifford deformations are always strongly $\mathbb{Z}_2$-graded and commutative.

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